A generating function for all-orders skyrmions

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We find the generating functions for the Lagrangians of all-orders summable $SU(2)$ Skyrmions. We then proceed to construct the explicit form of the Lagrangian, order by order in the derivatives of the pion field for two classes of models.

I. INTRODUCTION

According to the $1/N_c$ analysis [1,2], if an effective Lagrangian is to represent accurately the low-energy limit of QCD, it should behave as an effective theory of infinitely many mesons and include all orders of derivative of the fields. In that sense, the Skyrme model [3] represents a naive (fourth order in derivatives) attempt to provide such a description. Yet, stable solitonic pion field configurations (Skyrmions) emerge from this simple model. These solitons are interpreted as baryons and a number of their properties can be computed leading to many successful predictions. Unfortunately, the exact form of such a solution for the low-energy effective Lagrangian is out our reach for the moment and indeed, would be equivalent to a finding a solution for the low-energy limit of QCD and perhaps would provide an explanation to confinement. In the absence of such solutions, one must rely on more elaborate effective Lagrangians but in doing so one faces problems: (i) the number of possible extensions, i.e. the number of possible terms at higher orders becomes increasingly large, (ii) the degree of the equation of motion becomes arbitrarily large and (iii) adding terms of all orders introduces an arbitrarily large number of parameters, so one loses any predictive power. Any one of these problems causes the general approach to become too complex and its treatment intractable.

A suitable alternative is to impose symmetries or constraints that reduce this degree of arbitrariness and complexity but still retain the more interesting features and perhaps introduce new ones. A few years ago, we proposed such a class of tractable all-orders effective Lagrangians [4]: Using the hedgehog ansatz for static solution, we required that the degree of the differential equation for the profile function remains two. This induces all-orders Skyrmions with a number of interesting properties: (i) they come from all-orders effective Lagrangians that could account for infinitely many mesons, (ii) the Lagrangians being chirally invariant by construction, it allows more control over chiral symmetry breaking since they can be implemented by hand afterwards, (iii) there remains some liberty in this class of models so that the physical constraints can be satisfied properly and predictions can be significantly improved [5,6], (iv) their topological and stability properties are similar and (v) in some cases new interesting features such as a two-phase structure [7] arise.

Despite these relative successes, an all-orders Lagrangian written in a closed form remained to be found. Most of the calculations and constraints relied on an expression for the energy density of the static hedgehog solution. A full knowledge of the all-orders Lagrangian is required to analyze other types of solutions, generalize the model to $SU(3)$ and hopefully find a link to low-energy QCD. In this work, we construct the most general all-orders Lagrangians for the class of model introduced in [4] using a generating function. We then proceed to calculate the coefficients for the Lagrangian for any arbitrary order. This is done in Section III after a brief introduction of the all-orders Lagrangian in Section II. A similar procedure is repeated for effective Lagrangians induced by hidden gauge symmetry [8] and again, the coefficients for the Lagrangian are found for any arbitrary order in Section IV. This construction may have much deeper consequences since it is directly associated to an underlying gauge theory. Finally, the last section contains a brief discussion of the results and prospects for further analysis.

II. MORE TERMS TO THE SKYRME LAGRANGIAN

Let us recall the Skyrme Lagrangian density for zero pion mass:

$$\mathcal{L} = -\frac{F_\pi^2}{16}\,\mathrm{Tr}\,L_\mu L^\mu + \frac{1}{32e^2}\,\mathrm{Tr}\,f_{\mu\nu}f^{\mu\nu} \tag{1}$$

with the notation $L_\mu = U^\dagger \partial_\mu U$ and $f_{\mu\nu} \equiv [L_\mu, L_\nu]$. The first term, $\mathcal{L}_1$, coincides with the non-linear σ-model when one substitutes the degrees of freedom in the $SU(2)$ matrix U by σ- and π-fields according to $U = \frac{2}{F_\pi}(\sigma + i\tau \cdot \pi)$. The second term, $\mathcal{L}_2$, contains higher order derivatives in the pion field and can account for nucleon-nucleon interactions via pion exchange. $\mathcal{L}_2$ was originally added by Skyrme to allow for solitonic solutions. F_π is the pion decay constant (186 MeV) and e is the so-called Skyrme parameter. Unless F_π and e are explicitly mentionned, we shall use more appropriate units in which the Lagrangian rescale as

$$\mathcal{L}_1 + \frac{1}{2}\mathcal{L}_2 = \left(-\frac{1}{2}\,\mathrm{Tr}\, L_\mu L^\mu\right) + \frac{1}{2}\left(\frac{1}{16}\,\mathrm{Tr}\, f_{\mu\nu} f^{\mu\nu}\right) \tag{2}$$

and the unit of length is now $\frac{2\sqrt{2}}{eF_\pi}$ and the unit of energy is $\frac{F_\pi}{2\sqrt{2}e}$.

In its familiar hedgehog form, the $SU(2)$ matrix U is expressed as follows:

$$U(\mathbf{r}) = \exp\left[i\tau \cdot \widehat{\mathbf{r}} F(r)\right]$$

where $F(r)$ is called the chiral angle or profile function of the solution. This field configuration constitutes a map from physical space R^3 onto the group manifold $SU(2)$ and is assumed to go to the trivial vacuum for asymptotically large distances. We therefore impose $U(r \to \infty) \to 1$. From this last condition, one may derive the existence of a topological invariant associated with the mapping. The originality of Skyrme's idea was to identify this invariant, i.e. the winding number, with the baryon number.

For a static hedgehog solution, the energy density is given by $\mathcal{E} = \mathcal{E}_1 + \mathcal{E}_2$ where the fisrt contribution comes from the non-linear σ-model

$$\mathcal{E}_1 = -\mathcal{L}_1 = -\frac{1}{2}\,\mathrm{Tr}\, L_i L^i = [2a + b] \tag{3}$$

with $a \equiv \frac{\sin^2 F}{r^2}$ and $b \equiv F'^2$. Although $\mathcal{L}_2$ is quartic in the derivatives of the pion field, $\mathcal{E}_2$ adds only a quadratic contribution in $F^{'}$ to the Lagrangian for the static hedgehog solution.

$$\mathcal{E}_2 = -\mathcal{L}_2 = -\frac{1}{16}\,\mathrm{Tr}\, f_{ij} f^{ij} = a[a + 2b]. \tag{4}$$

Despite the relative successes of the Skyrme model, it can only be considered as a prototype of an effective theory of QCD. For example, there is no compelling reason or physical grounds for excluding higher order derivatives in the pion field from the effective Lagrangian. Indeed, large N_c analysis suggests that bosonization of QCD would most likely involve an infinite number of mesons. If this is the case, then taking the appropriate decoupling limits (or large mass limit) for higher spin mesons leads to an all-orders Lagrangian for pions. One example of higher order terms is a piece involving B_μ, the topological charge density [6],

$$\mathcal{L}_J = c_J \mathrm{Tr}\,\left[B^\mu B_\mu\right] = 3a^2 b \quad \text{with} \quad B_\mu = \epsilon_{\mu\nu\rho\sigma} L^\nu L^\rho L^\sigma$$

where c_J is a constant. It turns out that the term

$$\mathcal{E}_3 = -\mathcal{L}_3 = \frac{1}{32}\,\mathrm{Tr}\, f_{\mu\nu} f^{\nu\lambda} f_\lambda{}^{\mu} = 3a^2 b \tag{5}$$

leads to a similar results.

Several attempts were made to incorporate vector mesons in the Skyrme picture. Following this approach, we have proposed [5] a procedure to generalize the Lagrangian density to include all orders in the derivatives of the pion field in a computationally tractable way. In this work, we go a step further and will find the exact form of Lagrangian density at any arbitrary order.

At this point however, it is useful to invoke some relevant links noticed by Manton [9] between an effective $SU(2)$ scalar Lagrangian and the strain tensor in the theory of elasticity. As in nonlinear elasticity theory, the energy density of a Skyrme field depends on the local stretching associated with the map $U : R^3 \mapsto S^3$. This is related to the strain tensor at a point in R^3 which is defined as

$$\begin{aligned} M_{ij} &= \partial_i \Phi \partial_j \Phi \quad \text{where} \quad \Phi = (\sigma, \pi^z, \pi^x, \pi^y) \\ &= -\frac{1}{4} Tr[\{L_i, L_j\}] \end{aligned}$$

where i, j refers to the cartesian space coordinates. M_{ij} is a 3×3 symmetric matrix with three positive eigenvalues $\mathbf{X}^2, \mathbf{Y}^2, \mathbf{Z}^2$. The vectors $\mathbf{X}, \mathbf{Y}, \mathbf{Z}$ are orthogonal here since they are the principal axes of the strain ellipsoid in this context. There is a simple geometrical interpretation (due to Manton) for these objects. They correspond to the changes of length of the images of any orthogonal system in the space manifold. There are only three fundamental invariants of M_{ij} with each a simple geometric meaning

$$\begin{aligned} Tr[M] &= \mathbf{X}^2 + \mathbf{Y}^2 + \mathbf{Z}^2 = \sum (\text{length})^2 \\ \frac{1}{2}\left((Tr[M])^2 - Tr[M^2]\right) &= \mathbf{X}^2\mathbf{Y}^2 + \mathbf{Y}^2\mathbf{Z}^2 + \mathbf{Z}^2\mathbf{X}^2 = \sum (\text{surface})^2 \\ \det M &= \mathbf{X}^2\mathbf{Y}^2\mathbf{Z}^2 = (\text{volume})^2 \end{aligned}$$

Sometimes, it is convenient to use a more general ansatz [10] in which case the energy contributions to the first three orders take the form

$$\begin{aligned} \mathcal{E}_1 &= \mathbf{X}^2 + \mathbf{Y}^2 + \mathbf{Z}^2 \\ \mathcal{E}_2 &= (\mathbf{X} \times \mathbf{Y})^2 + (\mathbf{Y} \times \mathbf{Z})^2 + (\mathbf{Z} \times \mathbf{X})^2 \\ \mathcal{E}_3 &= 3\left(\mathbf{X} \cdot (\mathbf{Y} \times \mathbf{Z})\right)^2 \end{aligned} \tag{6}$$

Since these are the three fundamental invariants in an orthogonal system, all higher-order Lagrangians can be constructed out of these invariants. Another important quantity can also be written in term of these objects, the topological charge density

$$\mathcal{Q} = -\frac{1}{2\pi^2}\mathbf{X} \cdot (\mathbf{Y} \times \mathbf{Z})$$

For the hedgehog ansatz $\mathbf{X} \cdot \mathbf{Y} = \mathbf{Y} \cdot \mathbf{Z} = \mathbf{X} \cdot \mathbf{Z} = 0$ and $\mathbf{X}^2 = \mathbf{Z}^2$

$$a = \mathbf{X}^2 = \frac{\sin^2 F}{r^2} \quad b = \mathbf{Y}^2 = F'^2 \quad c = \mathbf{Z}^2 = \frac{\sin^2 F}{r^2} \tag{7}$$

and

$$\begin{aligned} \mathcal{E}_1 &= a + b + c = 2a + b \\ \mathcal{E}_2 &= ab + bc + ca = a(a + 2b) \\ \mathcal{E}_3 &= 3abc = 3a^2 b \end{aligned} \tag{8}$$

The angular integration is trivial in this case.

Let us consider the Lagrangian of an all-orders Skyrme-like model. In general, it contains even powers of the left-handed current L_μ, but in an orthogonal system, the static energy density is always a combination of the three invariants e.g. $\mathcal{E}_1, \mathcal{E}_2$ and $\mathcal{E}_3$. It turns out that one can construct a special class [4] of models whose energy density $\mathcal{E}$ is at most linear in b (or of degree two in derivatives of F). The static energy density coming from the Lagrangian of order $2m$ in derivatives of the field is of the form

$$\mathcal{E}_m = a^{m-1}[3a + m(b - a)] \tag{9}$$

for the hedgehog ansatz. The full Lagrangian leads to

$$\mathcal{E} = \sum_{m=1}^{\infty} h_m \mathcal{E}_m = 3\chi(a) + (b - a)\chi'(a)$$

where $\chi(x) = \sum_{m=1}^{\infty} h_m x^m$ and $\chi'(x) = \frac{d\chi}{dx}$ and to a profile equation which is computationally tractable since it is of degree two. In some cases, it is more appropriate to construct Lagrangians as m powers of the commutators $f_{\mu\nu} \equiv [L_\mu, L_\nu]$; this leads to vanishing energy density $\mathcal{E}_{2m}$ for m odd ≥ 5. Jackson et al [6] found an elegant expression for the total energy density in terms of the vectors $\mathbf{X}, \mathbf{Y}, \mathbf{Z}$

$$\mathcal{E}_J = \frac{(a - b)^3\chi(c) + (b - c)^3\chi(a) + (c - a)^3\chi(b)}{(a - b)(b - c)(c - a)} \tag{10}$$

where a, b, c are defined in (7). Note that both the numerator and the denominator are antisymmetric in a, b, c but the total expression is symmetric as should be expected since $\mathcal{E}$ is a combination of the three invariants $\mathcal{E}_1, \mathcal{E}_2$ and $\mathcal{E}_3$. As far as we know however, there seems to be no fundamental or geometric grounds that would justify such a form. Moreover, (10) is not very practical for our purposes since it cannot be easily converted to an expression in terms of the three invariants $\mathcal{E}_1, \mathcal{E}_2$ and $\mathcal{E}_3$.

The mass of the soliton is then written as:

$$M_S = 4\pi(\frac{F_\pi}{2\sqrt{2}e}) \int_0^\infty r^2 dr[3\chi(a) + (b-a)\chi'(a)]$$

Using the same notation, the chiral equation becomes:

$$0 = \chi'(a)[F'' + 2\frac{F'}{r} - 2\frac{\sin F \cos F}{r^2}] + a\chi''(a)[-2\frac{F'}{r} + F'^2 \frac{\cos F}{\sin F} + \frac{\sin F \cos F}{r^2}].$$

with $a \equiv \frac{\sin^2 F}{r^2}$. The Skyrme Lagrangian corresponds to the case $\chi(a) = \chi_S(a) \equiv a + \frac{1}{2}a^2$. In the absence of an exact solution for QCD that would provide a link to a Skyrme-like Lagrangian, one could consider the toy models in the form of an exponential or a truncated geometric series:

$$\chi_I(a) = e^a - 1$$

$$\chi_{II,M}(a) = a\frac{1-a^M}{1-a} = a + a^2 + a^3 + ... + a^M$$

which corresponds to the choice $h_{m \leq M} = \frac{(-)^{m-1}}{m}$. Yet, requiring that a unique soliton solution exists, $\chi(x)$ must satisfy

$$\frac{d}{dx}\chi(x) \geq 0, \quad x \geq 0$$
$$\frac{d}{dx}\left(\frac{\chi(x)}{x^3}\right) \leq 0, \quad x \geq 0$$
$$\frac{d}{dx}\left(\frac{1}{x^2}\frac{d}{dx}\chi(x)\right) \leq 0, \quad x \geq 0$$

which lead to more physically motivated alternative models due to Jackson et al [6] and to Gustaffson and Riska [7]:

$$\begin{aligned}
\chi_{III}(a) &= \ln(1+a) + \frac{1}{2}a^2 \\
\chi_{IV}(a) &= \frac{1}{4}[1 - e^{-2a}] + \frac{1}{2}a + \frac{1}{2}a^2 \\
\chi_V(a) &= a + \frac{a^3}{3+2a} \\
\chi_{VI}(a) &= a + \frac{a^3}{3+4a} + \frac{a^4}{1+4a^2} \\
\chi_{VII,M}(a) &= a + \frac{a^M}{M\sqrt{1+c_M a^{2M-6}}} \text{ with } c_M = \text{ constant.}
\end{aligned} \tag{11}$$

whose phenomenological implications can differ significantly from the original Skyrme model.

It should be emphasized that although this class of tractable Lagrangians was originally defined assuming the hedgehog ansatz (i.e. spherically symmetric solution), the same conditions apply for any orthogonal system (i.e. $\mathbf{X} \cdot \mathbf{Y} = \mathbf{Y} \cdot \mathbf{Z} = \mathbf{X} \cdot \mathbf{Z} = 0$) as long as any two of the three invariants are equal in which case one can write $a = \mathbf{Z}^2 = \mathbf{X}^2$, $b = \mathbf{Y}^2$ or $a = \mathbf{Y}^2 = \mathbf{Z}^2$, $b = \mathbf{X}^2$ or $a = \mathbf{X}^2 = \mathbf{Y}^2$, $b = \mathbf{Z}^2$. One can easily construct such a solution by making a conformal transformation on the hedgehog ansatz for example.

III. RECURSION RELATION AND GENERATING FUNCTION

As was pointed out by Jackson et al. in ref. [6], $\mathcal{E}_m$ is a function of the three invariants $\mathcal{E}_1, \mathcal{E}_2$ and $\mathcal{E}_3$ which obeys the recursion relation

$$\mathcal{E}_m = \mathcal{E}_{m-1}\mathcal{E}_1 - \mathcal{E}_{m-2}\mathcal{E}_2 + \frac{1}{3}\mathcal{E}_{m-3}\mathcal{E}_3 \tag{12}$$

Extending this result, it is easy to see that a similar relation holds for the Lagrangians (since $\mathcal{L}_m \to -\mathcal{E}_m$ in the static limit)

$$\mathcal{L}_m = -\mathcal{L}_{m-1}\mathcal{L}_1 + \mathcal{L}_{m-2}\mathcal{L}_2 - \frac{1}{3}\mathcal{L}_{m-3}\mathcal{L}_3 \tag{13}$$

At this point, we must stress on the importance of being able to construct the Lagrangian $\mathcal{L}_m$ at an arbitrary order $2m$. First the Lagrangian $\mathcal{L}_m$ is evidently a more fundamental object than the energy density $\mathcal{E}_m$ since it defines how the scalar fields interact with each other. Secondly, the recursion relation (12) only holds for the hedgehog ansatz but it is sufficient to impose the condition (13) on the Lagrangians. Finally, once the Lagrangian is known at any arbitrary order, it is possible to find other solitonic solutions, compute their time dependence and, generalize the model from $SU(2)$ to $SU(3)$ case or examine other extensions of the Skyrme model.

Let us rewrite the recursion relation (13) as

$$u_m = -u_{m-1}u_1 + u_{m-2}u_2 - \frac{1}{3}u_{m-3}u_3 \tag{14}$$

with $u_m = \mathcal{L}_m$ for any integer m.

By iteration, one would obtain

$$\begin{aligned} u_m &= -(-u_{m-2}u_1 + u_{m-3}u_2 - \frac{1}{3}u_{m-4}u_3)u_1 + u_{m-2}u_2 - \frac{1}{3}u_{m-3}u_3 \\ &= u_{m-2}(u_1^2 + u_2) + u_{m-3}(-u_1u_2 - \frac{1}{3}u_3) + u_{m-4}(\frac{1}{3}u_1u_3) \\ &= u_{m-3}(-u_1^3 - 2u_1u_2 - \frac{1}{3}u_3) + u_{m-4}(u_1^2u_2 + u_2^2 + \frac{1}{3}u_1u_3) + u_{m-5}(-\frac{1}{3}u_1^2u_3 - \frac{1}{3}u_3u_2) \\ &= ... \end{aligned} \tag{15}$$

and so on. There is several convenient ways to reformulate this recursion relation e.g. u_m can be rewritten as the following matrix operation:

$$u_m = \mathrm{Tr}[T^m S_0]$$

with

$$T = \begin{pmatrix} -u_1 & 1 & 0 \\ u_2 & 0 & 1 \\ -\frac{1}{3}u_3 & 0 & 0 \end{pmatrix} \quad \text{and} \quad S_0 = 3\begin{pmatrix} -1 & -\frac{u_2}{u_3} & -\frac{u_1}{u_3} \\ -u_1 & -\frac{u_1u_2}{u_3} & -\frac{u_1^2}{u_3} \\ u_2 & \frac{u_2^2}{u_3} & \frac{u_1u_2}{u_3} \end{pmatrix} \tag{16}$$

and the Lagrangian reads

$$\mathcal{L} = \sum_{m=1}^{\infty} h_m \mathcal{L}_m = \mathrm{Tr}[\chi(T)S_0]$$

But this iterative process is not practical for an arbitrary order m.

We need to find a closed form for

$$u_n = \sum_{n_1,n_2,n_3=0}^{\infty} C_{n_1,n_2,n_3} u_1^{n_1} u_2^{n_2} u_3^{n_3}$$

with $n = n_1 + 2n_2 + 3n_3$. For that purpose, we introduce the generating function

$$G(u_1,u_2,u_3;x) \equiv c_1 + \sum_{m=1}^{\infty} u_m x^m \tag{17}$$

where accordingly $u_m = \frac{1}{m!}\frac{d^m}{dx^m}G(u_1,u_2,u_3;x)\big|_{x=0}$. Here x is an auxiliary variable introduced for calculational purposes but one could interpret this variable as a scaling factor in the Skyrme model. Indeed, under a scale transformation $r \to \lambda r$, the Lagrangian u_m which is of order $2m$ in derivatives scales as

$$u_m \to \frac{u_m}{\lambda^{2m}} = u_m x^m$$

for $x = \lambda^{-2}$.

Using the recursion formula, it is easy to find an expression for G : In (17), we have

$$G = \sum_{m=4}^{\infty}\left(-u_{m-1}u_1 + u_{m-2}u_2 - \frac{1}{3}u_{m-3}u_3\right)x^m + u_3x^3 + u_2x^2 + u_1x + c_1$$

which upon substituting the summations by the generating function leads to

$$G = \frac{u_1^2x^2 + u_1xc_1 - u_2x^2c_1 + \frac{1}{3}x^3u_3c_1 + u_3x^3 + u_2x^2 + u_1x + c_1}{\left(1 + u_1x - u_2x^2 + \frac{1}{3}u_3x^3\right)}$$

This last identity is valid as long as the term in the numerator can be expanded in powers of u_1x, u_2x^2 and u_3x^3. Using the multinomial expansion

$$\left(1 + u_1x - u_2x^2 + \frac{1}{3}u_3x^3\right)^{-1} = \sum_{n_1,n_2,n_3=0}^{\infty}\frac{(n_1+n_2+n_3)!}{n_1!\ n_2!\ n_3!}\frac{(-)^{n_1+n_3}}{3^{n_3}}u_1^{n_1}u_2^{n_2}u_3^{n_3}x^{n_1+2n_2+3n_3}$$

one gets

$$G(u_1,u_2,u_3;x) = \sum_{n_1,n_2,n_3=0}^{\infty}\frac{(n_1+n_2+n_3)!}{n_1!\ n_2!\ n_3!}\frac{(-)^{n_1+n_3}}{3^{n_3}}\frac{\left(-4n_1n_3 + n_2^2 - n_2 - 2n_2n_3 - 3n_3^2 + 3n_3\right)}{(n_1+n_2+n_3)(n_1+n_2+n_3-1)}u_1^{n_1}u_2^{n_2}u_3^{n_3}x^{n_1+2n_2+3n_3}$$

This result is independent of the choice of c_1.

The next step consists in finding the explicit expression for u_m. This is acheived by isolating the term of order x^m in the previous expression. One finds for $m = n_1 + 2n_2 + 3n_3 \geq 4$

$$u_m = \sum_{n_2=0}^{[\frac{m}{2}]}\sum_{n_3=0}^{[\frac{m-2n_2}{3}]} C_{m-2n_2-3n_3,n_2,n_3}\ u_1^{m-2n_2-3n_3}\ u_2^{n_2}\ u_3^{n_3}$$

where $[z]$ stands for the integer part of z and

$$C_{n_1,n_2,n_3} = \frac{(n_1+n_2+n_3-2)!}{n_1!\ n_2!\ n_3!}\frac{(-)^{n_1+n_3}}{3^{n_3}}\left(-4n_1n_3 + n_2^2 - n_2 - 2n_2n_3 - 3n_3^2 + 3n_3\right)$$

The coefficients $C_{n_1n_2n_3}$ for $m = n_1 + 2n_2 + 3n_3 < 4$ are easy to find by inspection

$$C_{1,0,0} = C_{0,1,0} = C_{0,0,1} = 1$$
$$C_{0,0,0} = C_{2,0,0} = C_{3,0,0} = 0$$

Summing up, the full all-orders Lagrangian giving an energy density at most quadratic in F' has the form

$$\mathcal{L} = \sum_{m=1}^{\infty} h_m\mathcal{L}_m = \sum_{m=1}^{\infty}\sum_{n_2=0}^{[\frac{m}{2}]}\sum_{n_3=0}^{[\frac{m-2n_2}{3}]} h_mC_{m-2n_2-3n_3,n_2,n_3}\ \mathcal{L}_1^{m-2n_2-3n_3}\ \mathcal{L}_2^{n_2}\ \mathcal{L}_3^{n_3}. \tag{18}$$

IV. HIDDEN GAUGE SYMMETRY

In this section, we examine the construction of a generalized Skyrme Lagrangian based on the hidden gauge symmetry (HGS) formalism [8,4,11]. The procedure consist in introducing higher-order gauge terms to the Lagrangian to describe free vector mesons (for example $SU(2)$ gauge field kinetic Lagrangian $-\frac{1}{4}\,\mathrm{Tr}\,F_{\mu\nu}F^{\mu\nu}$) and in the substitution of the derivative by a covariant derivative to account for scalar-vector interactions. For $SU(2)$ chiral symmetry, the HGS formalism is based on the $SU(2)_L \otimes SU(2)_R \otimes SU(2)_V$ manifold where $SU(2)_V$ is gauged. The most general Lagrangian involving only two field derivatives is expressed as

$$\mathcal{L}_1^{HGS} = -\frac{F_\pi^2}{16}[Tr\left(L^\dagger D_\mu L - R^\dagger D_\mu R\right)^2 - \alpha Tr\left(L^\dagger D_\mu L + R^\dagger D_\mu R\right)^2] \tag{19}$$

where $L(x) \in SU(2)_L$ and $R(x) \in SU(2)_R$. D_μ stands for the covariant derivative $\partial_\mu - igV_\mu^k \cdot \frac{\tau^k}{2}$ where V_μ is the hidden gauge field. When the gauge vector field is dynamical then a second piece $\mathcal{L}_2^V = -\frac{1}{4}\,\mathrm{Tr}\,F_{\mu\nu}F^{\mu\nu}$ is added to the Lagrangian. The vector boson V acquires its mass from the same mechanism as the standard gauge bosons with result $m_V^2 = 4\alpha g^2 F_\pi^2$. In the large mass limit of the vector mesons, they decouple and an effective self-interaction for scalar mesons arise as $F_{\mu\nu} = [D_\mu, D_\nu] \to f_{\mu\nu} \equiv [L_\mu, L_\nu]$. Finally, in that limit, $\mathcal{L}_1^{HGS}$ becomes the non-linear σ-model $\mathcal{L}_1$ whereas $\mathcal{L}_2^V$ coincides with the Skyrme term $\mathcal{L}_2$.

Following this approach, we have proposed to represent contributions of order $2n$ in the derivatives of the pion field in terms of the trace of a product of n $f_{\mu\nu}$'s. Such terms would presumably come from gauge invariant quantities involving a similar expression with n field strengths $F_{\mu\nu}$ and would describe exchanges of higher spin mesons. For example, the lowest-order gauge invariant contributions may have the form

$$\begin{aligned}
\mathrm{Tr}\,F_{\mu\nu}F^{\mu\nu} &\to \mathrm{Tr}\,f_{\mu\nu}f^{\mu\nu} \\
\mathrm{Tr}\,F_\mu{}^\nu F_\nu{}^\lambda F_\lambda{}^\mu &\to \mathrm{Tr}\,f_\mu{}^\nu f_\nu{}^\lambda f_\lambda{}^\mu \\
\mathrm{Tr}\,(F_{\mu\nu}F^{\mu\nu})^2 &\to \mathrm{Tr}\,(f_{\mu\nu}f^{\mu\nu})^2 \\
\mathrm{Tr}\,F_\mu{}^\nu F_\nu{}^\lambda F_\lambda{}^\sigma F_\sigma{}^\mu &\to \mathrm{Tr}\,f_\mu{}^\nu f_\nu{}^\lambda f_\lambda{}^\sigma f_\sigma{}^\mu, \qquad etc...
\end{aligned}$$

The choice of such combinations is motivated by the possibility that they could be induced by hidden gauge symmetry (HGS) terms but they also correspond to exchanges of higher spin particles. They are automatically chirally invariant. Chiral symmetry breaking must be introduced independently, usually by adding a pion mass term, which means that in principle, one has more control on the symmetry breaking mechanism. As we can see from the first of the above expressions, the Skyrme term itself emerges from the gauge field kinetic term in the limit of large gauge vector mass in this formulation.

Furthermore, one can construct a special class [4] of such combinations which is at most linear in b (or of degree two in derivatives of F). These Lagrangians are a subset of those described in the previous section. They give a very simple form for the hedgehog energy density (similar to eq. (9))

$$\widetilde{\mathcal{E}}_{2m} = a^{2m-1}[3a + 2m(b-a)] \tag{20}$$

where m is an integer. It leads to a chiral angle equation which is tractable since it is of degree two.

It turns out that for this class of Lagrangians $\widetilde{\mathcal{E}}_m = 0$ for m odd ≥ 5 so we only need to consider Lagrangians which are of order $4m$ in derivatives of the pion field and have the form $\widetilde{\mathcal{L}}_{2m} \sim \mathrm{Tr}(f_{\mu\nu})^{2m}$. Although the constraints look similar, the Lagrangian $\widetilde{\mathcal{L}}_{2m}$ is different from a Lagrangian of the same order in derivatives $\mathcal{L}_{2m}$ described in the previous section since the latter being a combination of $\mathcal{L}_1 \sim Tr\,(L_i L_i)$ whereas $\widetilde{\mathcal{L}}_{2m}$ only involves $f_{\mu\nu}$'s. Yet it possible to write a recursion relation similar to (12) for the static energies

$$\widetilde{\mathcal{E}}_{2m} = \widetilde{\mathcal{E}}_{2m-2}\widetilde{\mathcal{E}}_2 - \widetilde{\mathcal{E}}_{2m-4}\widetilde{\mathcal{E}}_4 + \frac{1}{3}\widetilde{\mathcal{E}}_{2m-6}\widetilde{\mathcal{E}}_6$$

in terms of the three fundamental invariants $\widetilde{\mathcal{E}}_2, \widetilde{\mathcal{E}}_4$ and $\widetilde{\mathcal{E}}_6$. $\widetilde{\mathcal{E}}_{2m}$ arises from the Lagrangian $\widetilde{\mathcal{L}}_{2m}$ which obeys

$$\widetilde{\mathcal{L}}_{2m} = -\widetilde{\mathcal{L}}_{2m-2}\widetilde{\mathcal{L}}_2 + \widetilde{\mathcal{L}}_{2m-4}\widetilde{\mathcal{L}}_4 - \frac{1}{3}\widetilde{\mathcal{L}}_{2m-6}\widetilde{\mathcal{L}}_6$$

with

$$
\begin{aligned}
\widetilde{\mathcal{L}}_2 &= \frac{1}{16}\,\mathrm{Tr}\, f_{\mu\nu} f^{\mu\nu} \\
\widetilde{\mathcal{L}}_4 &= \frac{1}{64}\left(\mathrm{Tr}\, f_\mu{}^\nu f_\nu{}^\lambda f_\lambda{}^\sigma f_\sigma{}^\mu - \mathrm{Tr}\{f_\mu{}^\nu, f_\lambda{}^\sigma\} f_\nu{}^\lambda f_\sigma{}^\mu\right) \\
\widetilde{\mathcal{L}}_6 &= -\frac{1}{256}\left(\mathrm{Tr}\, f_\mu{}^\nu f_\nu{}^\lambda f_\lambda{}^\sigma f_\sigma{}^\rho f_\rho{}^\omega f_\omega{}^\mu - 2\,\mathrm{Tr}\{f_\mu{}^\nu, f_\lambda{}^\sigma\} f_\nu{}^\lambda f_\sigma{}^\rho f_\rho{}^\omega f_\omega{}^\mu + \frac{3}{2}\,\mathrm{Tr}\{f_\mu{}^\nu, f_\lambda{}^\sigma\}\{f_\nu{}^\lambda, f_\rho{}^\omega\} f_\sigma{}^\rho f_\omega{}^\mu\right).
\end{aligned}
$$

Again we are interested in a closed form for $\widetilde{\mathcal{L}}_{2m}$ at any arbitrary order, i.e.

$$
\widetilde{\mathcal{L}}_{2m} = \sum_{n_1,n_2,n_3=0}^{\infty} C_{n_1,n_2,n_3}\, \widetilde{\mathcal{L}}_2^{n_1}\, \widetilde{\mathcal{L}}_4^{n_2}\, \widetilde{\mathcal{L}}_6^{n_3}
$$

with $m = n_1 + 2n_2 + 3n_3$. Using the generating function technique, we see that the procedure is identical to that in the Section III upon the substitution $u_m = \widetilde{\mathcal{L}}_{2m}$ for any integer m. The full all-orders Lagrangian in derivatives of the pion field give rise to an energy density at most quadratic in F' and has the form

$$
\begin{aligned}
\widetilde{\mathcal{L}} &= \mathcal{L}_1 + \sum_{m=1}^{\infty} h_{2m} \widetilde{\mathcal{L}}_{2m} \qquad (21) \\
&= \mathcal{L}_1 + \sum_{m=1}^{\infty} \sum_{n_2=0}^{[\frac{m}{2}]} \sum_{n_3=0}^{[\frac{m-2n_2}{3}]} h_{2m} C_{m-2n_2-3n_3,n_2,n_3}\, \widetilde{\mathcal{L}}_2^{m-2n_2-3n_3}\, \widetilde{\mathcal{L}}_4^{n_2}\, \widetilde{\mathcal{L}}_6^{n_3}.
\end{aligned}
$$

where the first term is the nonlinear sigma model and the remaining of the expression accounts for the Skyrme term and higher-order terms. In this class of models, contributions of order $4m+2$ in derivatives of the pion field are absent.

Following the HGS formalism, this Lagrangian corresponds to the large mass limit of the vector mesons of a class of scalar gauged field theory described by the Lagrangian

$$
\mathcal{L} = \mathcal{L}_1^{HGS} + \sum_{m=1}^{\infty} h_{2m} \mathcal{L}_{2m}^V
$$

where $\mathcal{L}_{2m}^V$ is obtained from $\widetilde{\mathcal{L}}_{2m}$ upon substitution $f_{\mu\nu} \to F_{\mu\nu}$. The study of such theories and their justification based on physical grounds remains to be addressed.

V. CONCLUSION

Under their respective constraints, both (18) and (21) describe the most general all-orders Lagrangians. However some specific models are worth mentionning. For example, a simple choice of coefficients $h_m = 1$ corresponds to static energy densities

$$
\begin{aligned}
\chi(a) &= \frac{a}{1-a} = a + a^2 + a^3 + ... \\
\widetilde{\chi}(a) &= a + \frac{a^2}{1-a^2} = a + a^2 + a^4 + ...
\end{aligned}
$$

respectively and are induced by the Lagrangians written only in terms of the generating function

$$
\mathcal{L} = G(\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3; 1) = \frac{\mathcal{L}_1 + (\mathcal{L}_1^2 + \mathcal{L}_2) + \mathcal{L}_3}{\left(1 + \mathcal{L}_1 - \mathcal{L}_2 + \frac{1}{3}\mathcal{L}_3\right)} \qquad (22)
$$

$$
\widetilde{\mathcal{L}} = \mathcal{L}_1 + G(\mathcal{L}_2, \mathcal{L}_4, \mathcal{L}_6; 1) = \mathcal{L}_1 + \frac{\widetilde{\mathcal{L}}_2 + (\widetilde{\mathcal{L}}_2^2 + \widetilde{\mathcal{L}}_4) + \widetilde{\mathcal{L}}_6}{\left(1 + \widetilde{\mathcal{L}}_2 - \widetilde{\mathcal{L}}_4 + \frac{1}{3}\widetilde{\mathcal{L}}_6\right)} \qquad (23)
$$

In general, the term in the numerator of the generating function is not analytic so the expression must be understood as the correct analytic continuation of its series representation. In both cases, we chose $c_1 = 0$ since it only adds a constant piece to the full Lagrangian and otherwise would lead to an infinite energy solution. This result is easily

generalized to the model with coefficients $h_m = h^m$ since it is equivalent to a scale transformation and may be performed through the change of variables $a \to ha$, $\mathcal{L}_1 \to \mathcal{L}_1 h$, $\mathcal{L}_2 \to \mathcal{L}_2 h^2$, $\mathcal{L}_3 \to \mathcal{L}_3 h^3$ in the above expressions.

Let us now examine the Lagrangian in expression (22). For the hedgehog ansatz, we find an energy density

$$\mathcal{E} = -\frac{-\mathcal{E}_1 + (\mathcal{E}_1^2 - \mathcal{E}_2) - \mathcal{E}_3}{\left(1 - \mathcal{E}_1 + \mathcal{E}_2 - \frac{1}{3}\mathcal{E}_3\right)} \tag{24}$$

where $\mathcal{E}_1, \mathcal{E}_2$ and $\mathcal{E}_3$ are given by (8). This energy density $\mathcal{E}$ is obviously symmetric in a, b, c (for the hedgehog ansatz $a = c$) yet, it coincides with the energy density $\mathcal{E}_J$ given by the formula (10) of Jackson et al, i.e. the ratio of two antisymmetric expressions. For this rather simple model, it turns out to be possible to proceed backwards and deduce the full Lagrangian from $\mathcal{E}_J$: Starting from

$$\mathcal{E}_J = \frac{(a-b)^3\left(\frac{c}{1-c}\right) + (b-c)^3\left(\frac{a}{1-a}\right) + (c-a)^3\left(\frac{b}{1-b}\right)}{(a-b)(b-c)(c-a)}$$

and collecting terms which scale identically in the numerator and in the denominator, one can recast the energy density as

$$\mathcal{E}_J = -\frac{-(a+b+c) + (a^2 + ab + ca + b^2 + bc + c^2) - 3abc}{(1 - (a+b+c) + (ab + bc + ca) - abc)}$$

This is equivalent to (24) and suggest that the full Lagrangian has precisely the form in (22). A similar procedure can also be used for models where $\chi(a)$ is a rational function (quotient of two polynomials in a) e.g. $\chi_{II,M}(a), \chi_V(a)$ and $\chi_{VI}(a)$ in (11) but the complexity of calculations rises as the degree of the polynomials increases. On the other hand, these models are also easy to obtain in terms of the generating function G. More elaborate models requires the general expressions (18) and (21).

It is easy to show $\mathcal{E} = \mathcal{E}_J$ holds for any model represented by a function $\chi(x) = \sum_{m=1}^{\infty} h_m x^m$ in (10) using the most general form

$$\mathcal{E} = \sum_{m=1}^{\infty} \sum_{n_2=0}^{[\frac{m}{2}]} \sum_{n_3=0}^{[\frac{m-2n_2}{3}]} h_m C_{m-2n_2-3n_3, n_2, n_3} \, (-1)^{m-n_2+1} \mathcal{E}_1^{m-2n_2-3n_3} \, \mathcal{E}_2^{n_2} \, \mathcal{E}_3^{n_3}$$

and (8). Clearly, this result was to be expected since both constructions are based on the same constraints for the energy density. So besides finding the complete form of the most general all-orders Lagrangians, we have come up with a way to write the energy density in terms of polynomials symmetric in a, b, c.

In principle, it is possible to get an expression for the full Lagrangians written in term of the generating function G only, for any model in this class, e.g. models defined in (11). Note that in some cases, it may be more convenient to work with the *exponential generating function,* $E(x) = \sum_{m=1}^{\infty} \frac{1}{m!} u_m x^m$.

In summary, the generating function G was constructed for the class of models in which the energy density for the hedgehog ansatz is at most quadratic in F' and therefore computationally tractable. It led to an explicit expression for the coefficients C_{n_1,n_2,n_3} in (18) and (21). But now that the full Lagrangians are known, a number of questions remain to be addressed: (a) Solutions other than the $N = 1$ static hedgehog can now be analyzed thoroughly. For example, one can propose a different or a more general ansatz according to (6). (b) Also, solutions for $N > 1$ skyrmions should be examined either by a direct numerical calculation or a convenient ansatz (such as rational maps [12]). (c) It is now possible to write the time dependence for this class of models for which perhaps one could eventually provide a proper description.(d) The all-orders Lagrangians also allow to study analytically the identity map and to construct the mode spectrum in a closed form. Finally, the knowledge of this class of Lagrangians allows (e) to extend the models, for example in an extension from the $SU(2)$ to the $SU(3)$ symmetry group or (f) to use it in other areas, such as in weak skyrmions, baby skyrmions, etc...

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